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Note

On a question by Erdős about edge-magic graphs

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Abstract

A (p,q) graph is edge-magic if the vertices and edges can be labeled with distinct elements from the set $1,2,\ldots,p+q$ in such a way that the sum is the same along any edge. We give some general results about edge-magic graphs and show precisely which complete graphs are edge-magic.

Keywords: Edge-magic; Complete graphs

1. Introduction and general results

The question of whether or not a graph is edge-magic is one of a spate of labeling problems to receive recent attention. In a 1996 address, Ringel described edge-magic labelings of odd cycles, caterpillars, and stars, concluding these comments with the conjecture that even cycles and trees are also edge-magic [7,8]. Later in the same conference, Erdős queried 'What is the maximum number of edges in an edge-magic graph?' We will begin to answer Erdős' question by determining precisely which complete graphs are edge-magic.

A (p,q) graph is said to be *edge-magic* if the vertices and edges can each be labeled with distinct elements of the set $\{1,2,\ldots,p+q\}$ in such a way that the sum along any edge is the same. This sum is called the *magic-sum*, and is denoted by S. This is Ringel's definition of edge-magic, though the reader should be aware that alternative uses of the term appear in the literature (cf. [6]) and that edge-magic graphs were studied in 1970 by Kotzig and Rosa under the name *magic valuations*. At that time,

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they showed that cycles and complete bipartite graphs are edge-magic, thus proving Ringel's first conjecture before it was stated [4]. [1]

Let $T = p + q$ denote the largest label. Define an *edge-sum* to be the sum of the labels on an edge and its endpoints. Thus an edge-magic labeling is a one-to-one correspondence $\ell : (V \cup E) \to \{1, 2, \ldots, T\}$ for which the edge-sum is the same for every edge.

The following result shows a useful identity for edge-magic graphs. The identity given is the result of representing the sum of all edge-sums in two different ways.

Theorem 1. *An edge-magic graph satisfies the following equation*:

$$qS = \frac{T(T+1)}{2} + \sum_{i=1}^{p} (\deg v_i - 1) * \ell(v_i). \tag{1}$$

It is relatively straightforward to find graphs that are not edge-magic as can be seen in the following theorem.

Theorem 2. *If G is an r-regular (p,q) graph, $p \equiv 4 \bmod 8$ and r is odd, then G is not edge-magic.*

Proof. Suppose to the contrary that G is edge-magic and that we have an edge-magic labeling. We know that $pr = 2q$. So $q \equiv 2 \bmod 4$ and thus $T \equiv 2 \bmod 4$. Using Eq. (1) we have

$$qS = \frac{T(T+1)}{2} + (r-1) \sum_{i=1}^{p} \ell(v_i).$$

Observe that qS and $(r-1)\sum_{i=1}^{p} \ell(v_i)$ are even but that $T(T+1)/2$ is odd which is a contradiction. $\square$

Edge-magic labelings can be paired in a natural way as can be seen in the following theorem.

Theorem 3. *If G has an edge-magic labeling with sum S then G also has an edge-magic labeling with sum $3T + 3 - S$.*

Proof. Let G be an edge-magic graph with labeling ℓ and sum S. Relabel elements of G with a new labeling ℓ' where $\ell'(x) = (T+1) - \ell(x)$. First note that each label is

[1] In this paper, Kotzig and Rosa made a claim (without proof) that the only complete edge-magic graphs are K_1, K_2, K_3, K_5 and K_6. Their proof of this result is 'contained' in an unpublished technical report on complete graphs [5] which, in turn, depends on results in a previous 83 page technical report [3]. Our work regarding complete edge-magic graphs is self-contained and was done independently as we had neither access to nor, in fact, knowledge of these previous papers when we did the work.

used exactly once. Second if x, y, z were the original labels along a particular edge, the new edge-sum is $(3T+3)-(x+y+z)=3T+3-S$. □

We will call these labeling pairs *complements*.

2. Complete graphs

Erdős asked, 'What is the maximum number of edges that an edge-magic graph can have?' The first step in answering that question is to determine which complete graphs are edge-magic.

The graphs K_1 and K_2 are trivially edge-magic. The graph K_3 is a cycle and is therefore edge-magic [2]. One possible edge-magic labeling of K_3 has magic sum 9 and assigns the labels $\{1,2,3\}$ to the vertices. Theorem 2 shows that K_4 is not edge-magic. The graphs K_5 and K_6 are both edge-magic. One edge-magic labeling of K_5 has magic-sum 18 and assigns the labels $\{1,2,3,5,9\}$ to the vertices. One edge-magic labeling of K_6 has magic-sum 25 and assigns the labels $\{1,3,4,5,9,14\}$ to the vertices. The remainder of this section will show that no other complete graph is edge-magic [2].

Definition 4. Suppose a graph has an edge-magic labeling ℓ. Let the edge-difference of edge uv be defined as $|\ell(u)-\ell(v)|$.

Lemma 5. *Let G be a graph with edge-magic labeling ℓ. If H is a clique of G of order k with $r=\max(\{\ell(v)\colon v\in V(H)\}-\min\{\ell(v)\colon v\in V(H)\})$ then*

(1) *each edge of H corresponds to exactly one of the differences $1,2,\ldots,r$,*

(2) *nonadjacent edges in H must have unequal edge differences,*

(3) *an edge difference that occurs more than once in H must occur on exactly two adjacent edges,*

(4) *the differences $1,2,\ldots,\lfloor r/2\rfloor$ can occur at most twice and the differences $\lfloor r/2\rfloor+1,\ldots,r$ can occur at most once, and*

(5) $\binom{k}{2}\leqslant\lfloor 3r/2\rfloor$ *or equivalently* $r\geqslant\lceil k(k-1)/3\rceil$.

Proof. We first observe that (1) is obvious. To prove condition (2), suppose u_1v_1 and u_2v_2 are distinct edges with the same edge-difference $\delta=|\ell(u_1)-\ell(v_1)|=|\ell(u_2)-\ell(v_2)|$. Without loss of generality, assume $\ell(u_1)>\ell(v_1)$, $\ell(u_2)>\ell(v_2)$ and $\ell(u_1)>\ell(u_2)$. If u_1v_1 and u_2v_2 and are nonadjacent edges, then $\ell(u_1)+\ell(v_2)=\ell(u_2)+\ell(v_1)$. Since H is a clique, u_1v_2 and u_2v_1 and are edges of H. Also, since ℓ is an edge-magic labeling of G, $\ell(u_1)+\ell(v_2)+\ell(u_1v_2)=\ell(u_2)+\ell(v_1)+\ell(u_2v_1)$, so we have $\ell(u_1v_2)=\ell(u_2v_1)$. This establishes (2) by contradiction. Thus u_1v_1 and u_2v_2 are adjacent with common endpoint $v_1=u_2$, from which it follows that the edge difference of u_1v_2 is 2δ. As all

[2] It is incorrectly claimed in [1] that K_7 is edge-magic (see note in [2]).

edge-differences in H are bounded above by r, we have $\delta \leqslant r/2$. This shows that an edge-difference δ can be repeated only on adjacent edges and only if $\delta \leqslant r/2$. Since an equation of the form $|\ell(u) - n| = \delta$ has only two integer solutions n, it is not possible for three mutually adjacent edges to have the same edge-difference. Thus, (3) is established with (4) following immediately. From (1–4) we see that the size $\binom{k}{2}$ of H is bounded above by the number of available edge-differences, namely $(r - \lfloor r/2 \rfloor) + 2\lfloor r/2 \rfloor = \lfloor r/2 \rfloor + r = \lfloor 3r/2 \rfloor$. This establishes (5). □

We will make frequent use of (2) of the lemma, in particular that 'nonadjacent edges in H must have unequal edge-differences' and will refer to it by L2.

Theorem 6. *If p is an integer greater than or equal to* 7 then *K_p is not edge-magic.*

Proof. Suppose, for the purpose of contradiction, that there is a $p \geqslant 7$ for which K_p is edge-magic. Assume that this complete graph has an edge-magic labeling in which the vertex labels are $x_1, x_2, \ldots, x_p$ and the edge labels are $y_1, y_2, \ldots, y_q$, with both sequences in increasing order (where $q = \binom{p}{2}$). Let S denote the magic-sum for this labeling. Also, let T denote the largest label, namely $T = p + q = p + \binom{p}{2} = \binom{p+1}{2}$. We begin with the observation that, although we do not know the value of S, we can determine where the largest and smallest edge labels reside relative to the vertex labels. That is,

(1) $S = x_1 + x_2 + y_q$,

(2) $S = x_1 + x_3 + y_{q-1}$,

(3) $S = x_p + x_{p-1} + y_1$ and

(4) $S = x_p + x_{p-2} + y_2$.

From (1) and (2), we have

(5) $x_3 - x_2 = y_q - y_{q-1}$

and from (3) and (4), we have

(6) $x_{p-1} - x_{p-2} = y_2 - y_1$.

From (5) and (6), we can conclude that at least one of the labels $1, 2, T-1$, or T must belong to a vertex. Otherwise $x_3 - x_2 = y_q - y_{q-1} = x_{p-1} - x_{p-2} = y_2 - y_1$, which would violate L2. (Note that this violation depends on the fact that p is at least 6, making edges x_2x_3 and $x_{p-1}x_{p-2}$ nonadjacent.) Let us assume that

(7) $x_1 = 1$ or 2. (If 1 and 2 are both edge labels, then either $T-1$ or T is a vertex label. In this case, replace the current labeling with the complementary labeling described in Theorem 3, which has the assumed property).

Now, returning to our initial observations, from (1) and (3) we have

(8) $y_q - x_p = (x_{p-1} - x_2) + (y_1 - x_1)$.

This can be transformed using the following inequalities:

(9) $y_q \leqslant T$ by definition of T.

(10) $(y_1 - x_1) \geqslant -1$ since $y_1 \geqslant 1$ and $x_1 \leqslant 2$, and

Table 1
Cases for proof

Case	x_1	x_2	x_3	x_4	y_1	y_2	$x_{p-1}-x_{p-2}=$ y_2-y_1	Matching edge-difference
1	1	2	3	5	4	6	2	$x_4-x_3=2$
2	2	3	4	6	1	5	4	$x_4-x_1=4$
3	1	3	4	5	2	6	4	$x_4-x_1=4$
4	2	4	5	6	1	3	2	$x_4-x_2=2$
5	1	$d+1$	$d+2$	$d+3$	2	3	1	$x_4-x_3=1$
6	2	$d+2$	$d+3$	$d+4$	1	3	2	$x_4-x_2=2$

(11) $(x_{p-1}-x_2)\geqslant\lceil[(p-2)(p-3)]/3\rceil\geqslant 7$ where the first inequality follows from part (4) of Lemma 5 (using $V(H)=\{x_1,x_2,\ldots,x_{p-1}\}$) and the second from $p\geqslant 7$. Substitution of (9)–(11) into (8) yields

(12) $T-x_p\geqslant 6$. Thus, the largest six labels all correspond to edges. That is,

(13) $y_{q-j}=T-j$, for $j=0,1,\ldots,5$. This fact, together with the values of x_1 and x_2, force the values of x_3 and x_4. These, in turn, force the values of y_1 and y_2 and, via (6), the value of $x_{p-1}-x_{p-2}$. In each of the cases below, this forced value of $x_{p-1}-x_{p-2}$ matches an edge difference among the four smallest-labeled vertices, violating L2 and leaving us with our final contradiction. We can finish the proof by looking at six cases based on the values of x_1 and x_2.

We are let to consider several cases which are summarized in Table 1. These six cases correspond to the two possible values of x_1 (1 or 2) and the possible values of x_2-x_1 (1,2, or $d\geqslant 3$).

Case 1: If $x_1=1$ and $x_2=2$ then, by (1) and (13), $S=T+3$. By (2) and (13), $x_3=3$. Thus, $S=x_1+x_2+T=x_1+x_3+(T-1)=x_2+x_3+(T-2)$. Now we need to find two vertices whose labels sum to 6 to serve as endpoints for the edge labeled $T-3$. Since setting $x_4=4$ would violate L2, we must set $x_4=5$ to obtain $S=x_1+x_4+(T-3)$ and $y_1=4$. Since setting $x_5=6$ would violate L2, we know that $y_2=6$. Thus, by (6), $x_{p-1}-x_{p-2}=2$. However, since $x_4-x_2=2$ we again have a violation of L2.

Cases 2–6 are similiar and the details are shown in Table 1.

Note that all cases end in a violation of L2 as can be seen from the rightmost two colums of the table. These condradictions imply that no complete graph of order 7 or greater is edge-magic. □

3. Conclusions

We have shown the complete graphs, K_p, are not edge-magic when $p\geqslant 7$. Let $f(p)$ be defined as the maximum size among all edge-magic graphs of order p. Then, Erdős' question can be rephrased as 'Determine $f(p)$ for all natural numbers p'. It can be shown that K_4-e is edge-magic. In order to see this, take the following $v-e-v-\cdots-e-v$ labeling for C_4: $1,5,6,4,2,7,3,8$. Add an edge between the vertices labeled 1 and 2 and label this edge with the label 9. The result is an edge-magic labeling for $K_4 - e$. So

$$f(p) = \begin{cases} \binom{p}{2}, & p = 1, 2, 3, 5, 6 \\ 5, & p = 4. \end{cases}$$

Using the edge-magic graphs from this paper, we find some bounds for $f(p)$. Since complete graphs are not edge-magic when $p \geqslant 7$, $f(p) < \binom{p}{2}$. Kotzig and Rosa showed that complete bipartite graphs are edge-magic. The most edges in a complete bipartite graph of order p is $p^2/4$ if p is even and $(p^2 - 1)/4$ if p is odd. So, $(p^2 - 1)/4 \leqslant f(p) < \binom{p}{2}$, when $p \geqslant 7$.

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